

# Revisiting 5D chiral symmetry breaking and holographic QCD models

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## Abstract

We take a closer look at the recently discussed models of hadrons based on holographic ideas and chiral symmetry breaking in five dimensions. We study the two point correlator in detail and look at the field theoretic properties needed to be satisfied by the correlator. It is shown that the spectral density becomes negative, violating the basic requirements for the Kallen-Lehmann representation. We briefly discuss the implications of this violation of the positivity of the spectral density and also discuss possible origin of such a violation in these models. Put simply this means that such models are not physical descriptions of the hadron spectrum.

The correct and complete theory describing the confinement of quarks and gluons into hadrons, and explaining the hadronic properties from first principles was, and still is, one of the major challenges. Though there exist many phenomenological models which seem to describe the low energy data reasonably well, a complete theoretical understanding is still lacking. The holographic principle or the so called AdS/CFT correspondence [1] has opened a new window to look into the problem from a completely new and different perspective. Very loosely speaking, the correspondence is a conjecture of the equivalence of the generating functional of (large  $N$  limit of) certain conformal field theory (CFT) in  $d$ -dimensions and the  $(d + 1)$ -dimensional supergravity (string theory in a particular limit to be more precise, though we will use the word supergravity throughout the note) effective action evaluated at the boundary. The boundary value of the bulk fields is supposed to play the role of the source in the CFT generating functional. If we generically denote the  $(d + 1)$ -dimensional fields as  $\phi$ , then the correspondence can be schematically summarized as

$$\begin{aligned} Z_{AdS}[\phi_0] &= \int_{\phi_0} \mathcal{D}\phi \exp(-I[\phi]) \\ &\equiv Z_{CFT}[J = \phi_0] = \langle \exp(\int d^d x \mathcal{O} \phi_0) \rangle \end{aligned} \quad (1)$$

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where  $\phi_0$  is the boundary value of the field  $\phi$  which acts as a source corresponding to the operator  $\mathcal{O}$  in the CFT generating functional. For the  $(d+1)$ -dimensional AdS space, the line element is

$$ds^2 = \frac{L^2}{z^2}(dx_\mu dx^\mu + dz^2) \quad (2)$$

where  $x^\mu$  are the  $d$ -dimensional coordinates and  $z$  parameterizes the extra direction, with  $z=0$  describing the ultra-violet (UV) boundary and  $L$  denotes the radius of curvature, which we set to unity.

What it means in practical terms is that a perturbative calculation on the supergravity side can be translated into a non-perturbative result in the gauge theory sector and vice-versa. If this be so, it simply implies that the seemingly impossible task of getting the genuine non-perturbative results and calculating the non-perturbative quantities within a gauge theory is no more impossible. Instead, one can approach the problem from the dual side - where perturbative calculations, though hard, are possible. This has motivated considerable interest in getting the hadron spectrum and properties. Till very recently, most of the models constructed and studied can be broadly thought to fall under the *top-down* category of models i.e. the approach was to start from some string (or supergravity) theory and try to obtain the low energy description by demanding/imposing certain consistency conditions. For some of the earlier works, see [2]. Encouraged by these explorations, non-supersymmetric holographic dual models for hadrons have recently been proposed [3], [4]. Compared to previous studies, these models are phenomenological and the approach adopted is *bottom-up*. Guided by the basic ideas of the correspondence principle, one identifies the corresponding conserved currents which appear in the gauge theory as a dual description to the fields in the supergravity theory. This thus determines the (minimal) field content of the five dimensional theory of gravity, as dictated directly by the low energy sector. Thus the name, bottom-up. The models are still at the stage of being called *toy models* and they employ a very small sub-set of the possible field content. One can then go ahead and calculate various  $n$ -point Green functions (or correlators) of the fields in the 5-dimensional gravity theory. The correspondence relates such a calculation to the correlators of conserved currents in a suitable gauge theory. Using only a minimal sub-set of fields, the authors have shown that the models are quite robust and predictive to within 10% accuracy. To capture the essentials of chiral symmetry breaking, some specific boundary conditions are imposed on the fields and their derivatives on the so called “infra-red (IR) boundary” ( $z_{IR} \gg z_{UV}$ ). The theory is conformal only close to the UV boundary. As one moves away from this UV boundary, the theory is no more conformal. In a complete microscopic description, this should be incorporated by appropriately modifying the geometry which was AdS to start with. In the absence of a complete description, this is done by putting certain artificial and ad-hoc boundary conditions and one hopes to capture the essential and broad features.

In this Letter we take a closer look at these models and explore them in more detail. Given the robustness of the predictions within these models, it is tempting to investigate how far can they go in describing the experimental observations. The two point functions are the simplest and most straightforward to be evaluated within the 5D gravity theory. Their importance is not just being the simplest objects calculable, but lies in the fact that they offer a dual description to the current-current two point correlator  $\langle j_\mu j_\nu \rangle$ , in the gauge theory.

This correlator shows up in  $e^+e^-$  annihilating into hadrons,  $\tau$  hadronic decays, hadronic contribution to  $(g-2)_\mu$  and many other places. Moreover, the current-current correlator is used to extract the masses and decay constants of the mesons in the large  $N_c$  limit of QCD. Furthermore, it is known that the masses and decay constants obey certain sum rules, namely the Weinberg sum rules and generalizations [5]. It is thus important to have a precise and accurate theoretical description of the two point current correlation function.

Before proceeding to explore the two point function in more detail, let us briefly mention the field content and other details of the *toy model* [4] that we will be working with. We follow [4] for the notation and general setup of the model. The field content of the 5D theory is dictated by the operators in the 4D QCD. In principle, there should be an infinite number of fields corresponding to an infinite number of gauge invariant operators in QCD. However, for the purpose of chiral symmetry breaking and its essential consequences, it suffices to look at a very small number of operators and therefore a small number of fields in the 5D theory. As is known, the chiral dynamics is quite effectively described by an  $SU(N_f)_L \times SU(N_f)_R$  theory. We therefore have the following three operators that are most crucial to this effect (corresponding 5D fields are written in front of each of them):

$$\begin{aligned}\bar{q}_L(x)\gamma^\mu T^a q_L(x) &\xrightarrow{dual} A_L^{a\mu}(x, z) \\ \bar{q}_R(x)\gamma^\mu T^a q_R(x) &\xrightarrow{dual} A_R^{a\mu}(x, z) \\ \bar{q}_L^i(x)q_R^j(x) &\xrightarrow{dual} (2/z)X^{ij}(x, z)\end{aligned}\tag{3}$$

where  $T^a$  are the group generators and  $i, j$  are the flavour indices. Restricting to two flavours implies that  $i, j = 1, 2$  and the  $T^a$ 's are the three Pauli matrices. We are thus looking at a chiral  $SU(2)_L \times SU(2)_R$  theory. With this minimal field content and ignoring any interactions for the time being, the 5D action is

$$S_{(5)} = \int d^4x dz \sqrt{g} \text{Tr} \left[ -\frac{1}{4g_5^2} (F_L^2 + F_R^2) + |DX|^2 - m^2 X^2 \right] \tag{4}$$

where  $F_{L,R}^{AB}$  denotes the field strengths for the left and the right gauge fields and  $D_A$  denotes the covariant derivative.

At the IR boundary, some boundary conditions are imposed on the fields. Moreover, the reason behind introducing the IR boundary in the theory is to parameterize the effect of chiral symmetry breaking in an effective fashion. Else, one would be forced to examine how the geometry of the bulk changes away from the UV boundary and what should be the metric far away that will suitably describe chiral symmetry breaking in the 4D gauge theory. The introduction of the IR boundary and suitable boundary terms in a phenomenological way takes care of this aspect. For the gauge fields, we impose the boundary conditions:  $F_{L,R}^{\mu z} = 0$  and choose to work in the gauge  $A_z = 0$ . For the field  $X$ , the expectation value at the UV boundary is the quark mass matrix and the quark condensate will effectively fix the other constant (at the IR boundary) in the solution to the equation of motion for field  $X$ . In principle, the UV boundary corresponds to  $z = 0$ . However, in practice, the boundary conditions are specified at  $z = z_{UV}$  and finally the limit  $z_{UV} \rightarrow 0$  is taken.

Define the vector and axial-vector gauge fields as appropriate linear combinations of  $A_L$  and  $A_R$ . Let us focus on the vector gauge field  $V_A = \frac{1}{2}((A_L)_A + (A_R)_A)$  and let  $\tilde{V}_A(q, z)$  denote the Fourier transform of the vector field with respect to the 4D coordinates. We have suppressed the group index for convenience. In the  $V_z = 0$  gauge, the linearized equation of motion for the transverse part reads

$$\partial_z \left( \frac{1}{z} \partial_z \tilde{V}_\mu(q, z) \right) + \frac{q^2}{z} \tilde{V}_\mu(q, z) = 0 \quad (5)$$

The solution is a linear combination of the Bessel functions and can be written as

$$\tilde{V}_\mu(q, z) = C_\mu(q) q z [b \mathcal{J}_1(qz) + \mathcal{Y}_1(qz)] \quad (6)$$

The boundary condition at the IR boundary, namely  $\partial_z V_\mu|_{z=z_{IR}} = 0$ , fixes the relative constant between the two terms to be

$$b \approx -\frac{\mathcal{Y}_0(qz_{IR})}{\mathcal{J}_0(qz_{IR})} \quad (7)$$

We can now use the correspondence principle to interpret the above solution at  $z = z_{UV}$  as the source for a current. For the two point function we get [6],

$$\langle j_\mu(0) j_\nu(q) \rangle = \left( \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \Pi(q) \quad (8)$$

where

$$\Pi(q)|_{z=z_{UV}} = \left( \frac{1}{g_5^2} \right) \frac{q}{z_{UV}} \left[ \frac{\mathcal{Y}_0(qz_{UV}) + b \mathcal{J}_0(qz_{UV})}{\mathcal{Y}_1(qz_{UV}) + b \mathcal{J}_1(qz_{UV})} \right] \quad (9)$$

This expression can be expanded about  $z_{UV} = 0$  and we retain the leading terms, which leads to

$$\Pi(q)|_{z=z_{UV}} \approx \left( \frac{1}{g_5^2} \right) q^2 \left[ \log(qz_{UV}/2) + \gamma_E + \frac{\pi}{2} b \right] \quad (10)$$

The logarithmic term may be identified as the contribution to the current-current correlator arising from the lowest order quark bubble diagram. Comparing this expression with the one obtained within QCD with  $N_c$  colours fixes the 5D gauge coupling,  $g_5$ , in terms of  $N_c$ .

The next task is to estimate the hadron masses and decay constants. This is easily done by comparing the two point function obtained above with the corresponding expression that one obtains in the large  $N_c$  chiral theories. In such a theory, the two point function is expressed as a sum over narrow resonances. The poles of the two point function yield hadron masses and the residues at each pole are the decay constants. The correlator thus has the following form (the  $i\epsilon$  is implicit)

$$\Pi(q^2) = q^2 \sum_n \frac{F_n^2}{q^2 - M_n^2} \quad (11)$$

In particular, the above form and the fact that we have approximated the correlator as a sum over narrow resonances imply that the spectral density is a *comb of delta functions* peaking at the hadron masses. The spectral density is nothing but the imaginary part of  $\Pi(q^2)$ . Also,

let us recall that the Kallen-Lehmann representation for the two point function in a generic field theory implies that the spectral density is a positive quantity [7]. This is an important property that it must satisfy and we'll see that this property plays an important role in our analysis.

To estimate the hadron masses, we look for the poles of the two point function. We can use either Eq(9) or Eq(10) for this purpose. The poles are given by the zeros of  $\mathcal{J}_0(qz_{IR})$ . Then using the experimental value for  $m_\rho$  as an input fixes  $z_{IR}$ . The decay constants are the residues at these poles.

The estimated numbers for the masses and decay constants [4] compare well with the experimental values and we get the impression that the model is very robust and quite close to reality. However, as we'll see below, this is not the real and complete picture. We take a closer look at the two point function. In particular, we are interested in the corresponding spectral density,  $\rho(q^2)$ . Both the vector and axial-vector spectral densities are measured to very good accuracy, for example, in the hadronic  $\tau$  decays and are directly related to the decay width by optical theorem [8]. The data clearly shows the  $\rho$  and  $a_1$  resonance peaks and supports the theoretical expectation of the spectral densities to be positive functions. We use Eq(10) (or equivalently Eq(9)) to extract the imaginary part, and therefore the spectral density. Using the usual  $i\epsilon$  prescription, it is easy to obtain the imaginary part which is nothing but the following

$$\rho(q) = Im \left[ \frac{\mathcal{Y}_0(qz_{IR})}{\mathcal{J}_0((q+i\epsilon)z_{IR})} \right] \quad (12)$$

We would like to study the above spectral density in more detail and convince ourselves that it satisfies all the field theoretic properties, like the positivity condition, before we can proceed further and calculate masses and decay constants from the two point correlation function. To this end, we consider the following function

$$f(x) = \frac{\mathcal{Y}_0(x)}{\mathcal{J}_0(x+i\epsilon)} = Re[f(x)] + iIm[f(x)] \quad (13)$$

Clearly, the imaginary part of the function is nothing but the spectral density itself rewritten in terms of a different variable. In Figure 1, we have plotted the imaginary part of the function,  $Im[f(x)]$ .<sup>1</sup> A quick look at the plot gives the impression that the spectral density is indeed a comb of delta functions, as expected and desired. The zeros of the Bessel function  $\mathcal{J}_0(x)$  are the positions of the resonance masses and the residues at these values will correspond to the decay constants of the mesons. Let us now take a more closer look at the function itself. The function under investigation is a ratio of two Bessel functions. Further, the Bessel functions are known to have an oscillatory behaviour, with  $\mathcal{J}_0(x)$  and  $\mathcal{Y}_0(x)$  having opposite behaviour with respect to each other. Figure 2 is a plot of the two Bessel functions, clearly showing these features.

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<sup>1</sup>A quick way to see that the imaginary part will have a profile very similar to a sum of delta functions is to look at the series expansion,  $\mathcal{J}_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \dots$ . The  $i\epsilon$  prescription can now be used to obtain the imaginary part. For plotting the graph, we have chosen  $\epsilon = 10^{-7}$  and rescaled the y-axis. It should be borne in mind that any other small value of  $\epsilon$  is equally good and rescaling simply helps in having an enlarged picture and does not change the shape and nature of the curve.

The relevant quantity (the imaginary part of the function  $f(x)$  which is the spectral density) is a ratio of these two different Bessel functions and because of the features mentioned above,  $\mathcal{Y}_0(x)$  can cross the real axis between two zeros of  $\mathcal{J}_0(x)$ , thereby yielding negative values for the imaginary part of the function, and therefore the spectral density. To further substantiate our claim, we look at the behaviour of the imaginary part of the function  $f(x)$  more closely. In Figure 3, we plot  $Im[f(x)]$  for various smaller intervals of  $x$  and show that it indeed acquires negative values. The reason that this feature is not evident in Figure 1 is due to the very large values that the function acquires close to the resonances. However, when we look at the behaviour in regions slightly away from the resonances, the negative values show up, as shown in Figure 3.

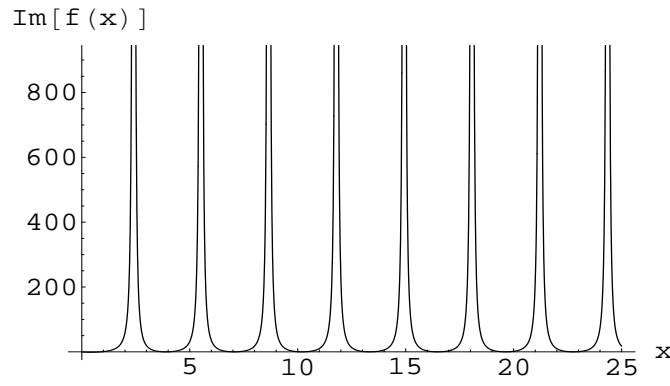


Figure 1: *Coarse-grain* view of  $Im[f(x)]$  as a function of the argument,  $x$

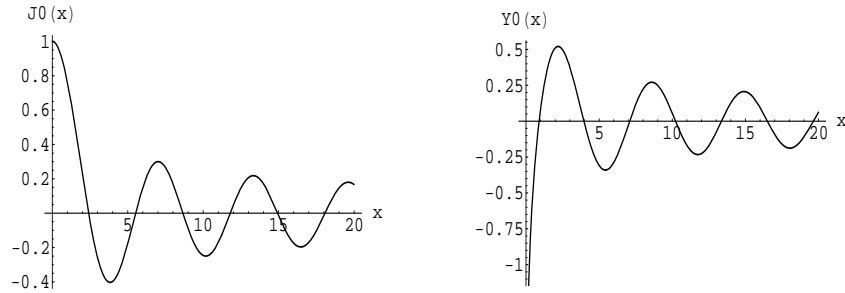


Figure 2: The two Bessel functions:  $\mathcal{J}_0(x)$  (left) and  $\mathcal{Y}_0(x)$  (right)

Let us put all the individual pieces of information together to get a final and complete picture. We have seen above that a cursory look at the imaginary part gives the impression that it is a comb or sum of delta functions - the position of the peaks will give the masses of the mesons and the residues at these values will be the respective decay constants. It is quite clear from the Figure 1 that the residues at the peak positions of the imaginary part are indeed positive. It is this *coarse-grain* picture that leads us to believe that all is well with the model and the predicted numerical values for the meson masses and decay constants are in agreement with the experimental values to within 10 – 20%. However, as is clear from Figure 3, the imaginary part of the function (the spectral density) acquires negative values. Also, it is clear from the figure that we encounter negative values in a somewhat periodic manner

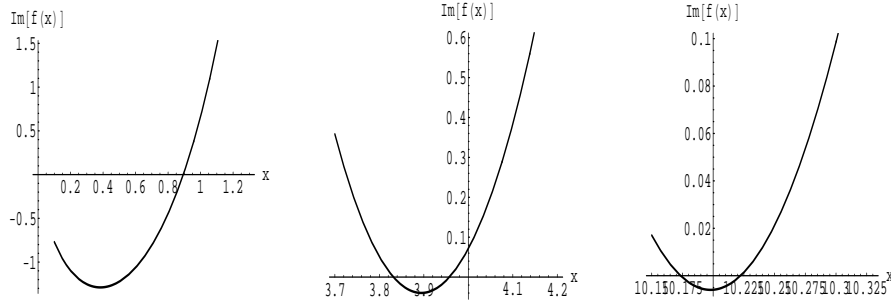


Figure 3:  $Im[f(x)]$  plotted in different intervals

and that the magnitude of the negative values attained diminishes as we go to larger values of the argument. This feature is also not hard to expect and understand. The “somewhat periodic” appearance of the negative values is due to the oscillatory behaviour of Bessel functions (or their combination) while from the behaviour of the Bessel functions, it is very evident that the amplitude keeps on decreasing for larger and larger values of the argument.

It is now straightforward to convince ourselves that the spectral density obtained within these models is not positive semi-definite. However, as we have discussed earlier, the spectral density is a positive quantity and has been very well measured in experiments, in full conformity with our theoretical expectations. This is the most important observation and result of this study. It may be worthwhile to briefly comment on the results/findings of the models [3, 4] in the light of this observation. As has been mentioned above, if we just content ourselves with the positions of the resonances (meson masses) and evaluate the residues at these positions (decay constants), we’ll obtain positive values for them. This is essentially what is done in [3, 4]. It is only when we take a closer look at the function under consideration that we are led to the observation that the imaginary part of the function, which is supposed to be positive always, acquires negative values.

The above observation regarding the spectral density becoming negative is a serious issue of concern. As was mentioned, the spectral density is directly related to the hadronic  $\tau$  decay width by the optical theorem. A negative spectral density will simply mean that the decay width is becoming negative - something that is clearly unphysical and of course unobserved. Translated back, this observation has to say something very important about the model itself. In this form, the model does not lead to physical predictions and therefore can not be trusted. Similar conclusions can be reached at from the data on  $e^+e^-$  annihilating into hadrons or the hadronic contributions to  $(g-2)_\mu$ . In each case the spectral density is related to a positive physical observable like cross-section or decay rate.

We make a brief attempt to discuss the possible origin of such a problem in these models. Recall that to capture the essential features of chiral symmetry breaking, an ad-hoc and artificial infra-red boundary was introduced in the theory and certain specific boundary conditions specified on it. This approach is completely phenomenological and though, intuitively may seem well motivated and correct, by itself, does not guarantee that the results will be unitary and physical. As was pointed out initially, in a complete microscopic description, the geometry of the space-time should be appropriately modified and this should be consis-

tently done so that away from the AdS boundary ( $z = 0$ ), the model incorporates the correct pattern for chiral dynamics. However, in the case at hand, this was avoided by invoking the artificial IR boundary. In our opinion, this itself is the root cause of the problem. The reason is as follows. Both the masses and the decay constants are obtained by essentially looking at the constant " $b$ " that appears in the solution to the equation of motion for the gauge field. This constant is fixed in the present scenario by requiring the derivative of the solution to vanish at the artificial IR boundary, thus yielding a ratio of two Bessel functions. It is this combination of Bessel functions that leads to the trouble. We may be led to speculate that if, instead of the approximate form for the relative constant  $b$  in Eq(7), we had used the full expression, we would have bypassed the problem. It is again easy and straightforward to convince ourselves that this is not the case as the new form is also a combination of some other Bessel functions. If however, we can model the chiral symmetry breaking pattern by continuously changing the geometry in the bulk, this problem can possibly be avoided. We would like to point out that this is a common problem in all the models that invoke an IR boundary condition to model the chiral symmetry breaking. We restricted ourselves to the vector sector of the theory but the same arguments and conclusions apply to the axial vector sector as well. One can also check that the Weinberg sum rules are not satisfied in these models and there is no explanation of the (approximate)  $\rho$  meson dominance that is observed in nature.

We conclude by summarizing our main observation and some of its consequences. We have investigated the profile of the spectral density in the recently proposed holographic models of QCD or chiral symmetry breaking in five dimensions [3, 4]. The models seem to be quite robust and naively taken, seem to be predictive to within 10% accuracy. However, a closer look reveals that the spectral density, extracted from the two point correlation function, keeps acquiring negative values. This is in contrast to the positive behaviour of the spectral density expected from very general field theory arguments. Moreover, the vector and the axial-vector spectral densities are directly measured, for example, in the corresponding hadronic  $\tau$  decay modes. A negative spectral density implies a negative decay rate, in clear violation with unitarity and optical theorem and also with the observed data. This simply implies that the proposed models, though seem remarkably predictive, do not satisfy some of the basic field theoretic requirements and therefore can not be trusted. Also, in this form they can not be seriously taken to be models describing the physical hadron spectrum. Similar arguments will hold for any other model that violates the positivity condition for the spectral density and/or is in conflict with generic field theoretic expectations. As pointed out, the root cause in the present case is the way chiral symmetry breaking has been modeled by introducing an ad-hoc IR boundary.<sup>2</sup> If on the other hand, this is done in a more consistent manner by suitably modifying the bulk geometry in a continuous fashion, there is hope to get a dual model of hadrons which avoids the above mentioned problem.

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<sup>2</sup>This feature is expected to be present in any model that invokes an artificial IR boundary condition to mimic chiral symmetry breaking. We would also like to mention that the observations of the present study need not apply to the works listed in [2].



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